Expected Values
Averages
Linearity

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Proof Linearity

Let $T = X + Y$,

Show $E(T) = E(X) + E(Y)$
(so long as expectations exist)
Clear when X and Y are independent, not when they are dependent.

Discrete Case

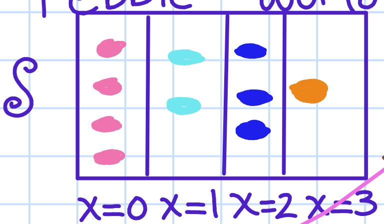
definition expectation

$$\sum_t t\,P(T=t)$$

$$\stackrel{?}{=} \sum_x x\,P(X=x) + \sum_y y\,P(Y=y)$$

$$P(T=t) = \sum_x P(T=t \mid X=x)\,P(X=x)$$

Pebble World



grouped

$$E(X) = \sum_x x\,P(X=x)$$

Instead of summing over x, we can sum over all pebbles in S.

$$= \sum_s X(s)\,P(s)$$

ungrouped

mass of pebble s

Proof of Linearity
Rewrite expectation

$$E(T) = \sum_s (X+Y)(s)\,P(\{s\})$$

$$= \sum_s (X(s)+Y(s))\,P(\{s\})$$

$$= \sum_s X(s)\,P(\{s\}) + \sum_s Y(s)\,P(\{s\})$$

$$= E(X) + E(Y)$$

$E(cX) = cE(X)$ if c is constant.

Important Discrete Distribution

Negative Binomial (Don't be fooled)

Parameters r, p

Story: independent Bern(p) trials, # failures before the rth success.

PMF: $\boxed{1000100100001001}$ Ends in a 1.

Let $r = 5$, $n = 11$

If we permute the 0's and 1's before the last 1, it will not change things.

PMF $\boxed{P(X=n) = \binom{n+r-1}{r-1} p^r (1-p)^n}$

$n = 0, 1, 2, 3, \ldots$

Find $E(X)$

Think of simple and extreme cases

If $r=1$, we have the geometric, which has expected value q/p.

If $r=2$, wait for first success, then for the second success.

$$E(X) = E(X_1 + \cdots + X_r)$$

where X_j is # failures between $(j-1)$ and jth success.

$$X_j \sim \text{Geom}(p)$$

X_js are independent

by Linearity

$$= E(X_1) + \cdots + E(X_r)$$

$$= \boxed{rq/p}$$

First success distribution

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Let $X \sim FS(p)$ time until 1st success.

Let $Y = X-1$. Then $Y \sim Geom(p)$

$$E(X) = E(Y+1) = E(Y) + 1 = \frac{q}{p} + 1 = \boxed{\frac{1}{p}}$$

Problem from Putnam Exam

Random permutation of $1, 2, \ldots, 3$, $n \geq 2$. Find the average or expected number of local maxima.

Example

3 2 1 4 7 5 6   3 local maxima

Let I_j be indicator r.v. of position j having a local max.

Sum I_js is the number of local maxima.

$E(I_1 + \cdots + I_n)$ by Linearity

$= E(I_1) + \cdots + E(I_n)$

$\underbrace{475}$

$1/3$ chance the larger # is in the middle.

We can also take advantage of the symmetry of the problem.

There are $n-2$ intermediate points

$$E(I_1) + \cdots + E(I_n) = \frac{n-2}{3} + \frac{1}{2} + \frac{1}{2}$$

$$= \boxed{\frac{n+1}{3}}$$

St Petersburg Paradox

Get $\$2^X$ where X is #flips of fair coin until first head, including the success.

Let $Y = 2^X$, Find $E(Y)$

$$E(Y) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

bound at $\$2^{40}$. Then

$$\sum_{k=}^{40} 2^k \cdot \frac{1}{2^k} = 40$$

$$E(2^X) = \infty \neq 2^{E(X)} = 4$$