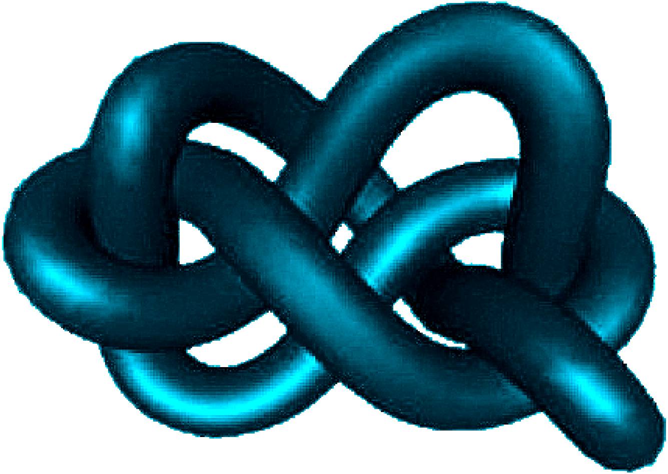
Hard Unknots & Collapsing Tangles
Louis Kauffman and Sofia Lambropoulou

Crista Moreno
Mathematics Department
San Francisco State University

March 15, 2013

Is this knot the unknot?

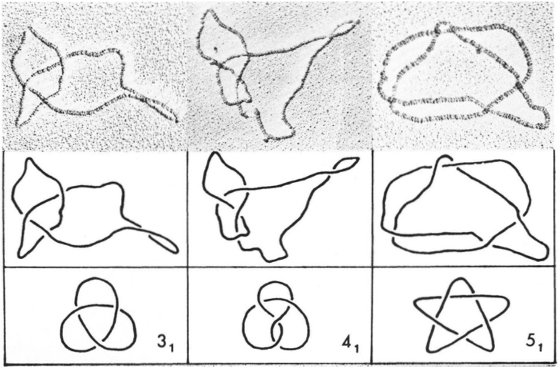


Unknot Theorem

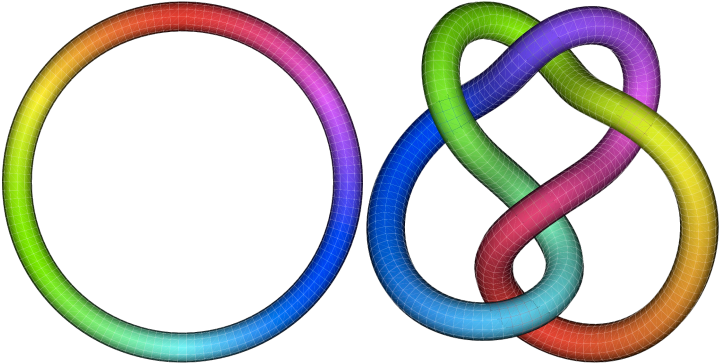
Theorem

Let $\dfrac{P}{Q} = [a_1, \ldots, a_n]$ *and* $\dfrac{R}{S} = [b_1, \ldots, b_n]$ *be as in Theorem 5. Then* $N\left(\left[\dfrac{P}{Q}\right] + \left[\dfrac{R}{S}\right]\right)$ *is unknotted if and only if* $PS + QR = \pm 1$*, that is,* PS *and* QR *are consecutive integers.*

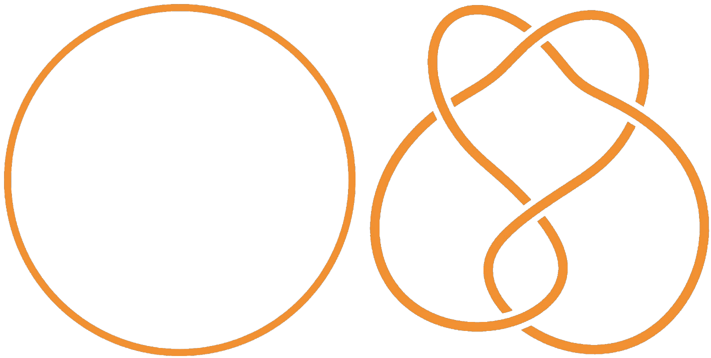
Biological Motivation



Dean, F. B., Stasiak, A., Koller, T. & Cozzarelli, N. R. (1985) J. Biol. Chem. 260, 4975-4983.

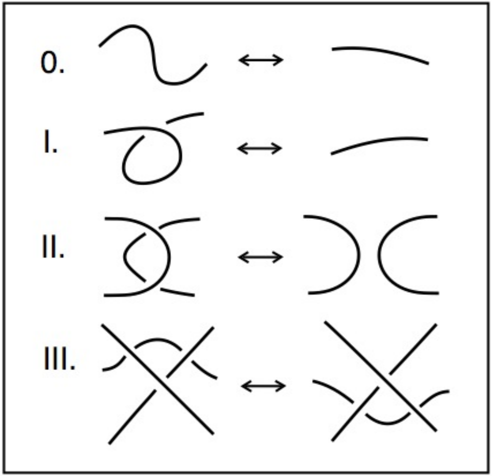


How to determine if two knots are equivalent?



How to determine if two knot diagrams are equivalent?

Reidemeister Moves



Definition Hard Unknot

Definition Hard Unknot

A diagram of the unknot **hard** if it has the following three properties, where a move is simplifying if it reduces the crossing number of the diagram:

- There are no simplifying type I or type II moves on the diagram.
- There are no type III moves on the diagram.

Definition Hard Unknot

A diagram of the unknot **hard** if it has the following three properties, where a move is simplifying if it reduces the crossing number of the diagram:

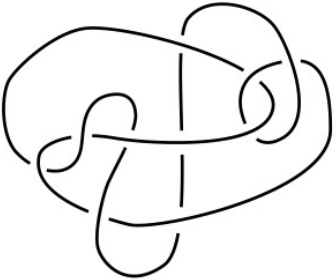
- There are no simplifying type I or type II moves on the diagram.
- There are no type III moves on the diagram.

Definition Hard Unknot

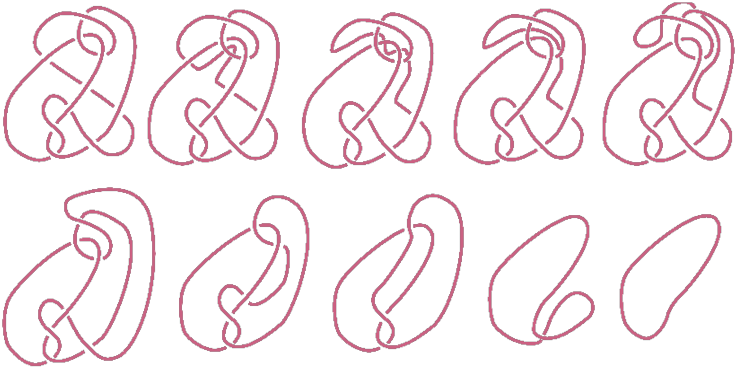
A diagram of the unknot **hard** if it has the following three properties, where a move is simplifying if it reduces the crossing number of the diagram:

- There are no simplifying type I or type II moves on the diagram.
- There are no type III moves on the diagram.

Example: Ken Millet's "The Culprit"



The Culprit Undone



Definition Recalcitrance

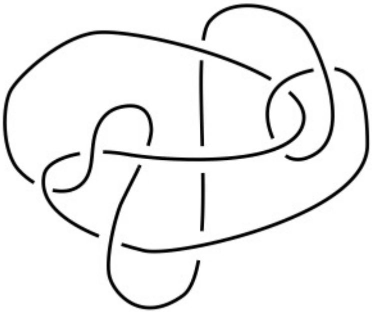
Definition Recalcitrance

Recall that the complexity of a diagram K is the number of crossings, $C(K)$, of that diagram. Let K be a hard unknot diagram. Let K' be a diagram isotopic to K such that K' can be simplified to the unknot. For any unknotting sequence of Reidemeister moves for K there will be a diagram $K'max$ with a maximal number of crossings. Let $Top(K)$ denote the minimum of $C(K'max)$ over all unknotting sequences for K. Let

$$R(K) = \frac{Top(K)}{C(K)} \tag{1}$$

be called the **recalcitrance** of the hard unknot diagram K. Very little is known about $R(K)$.

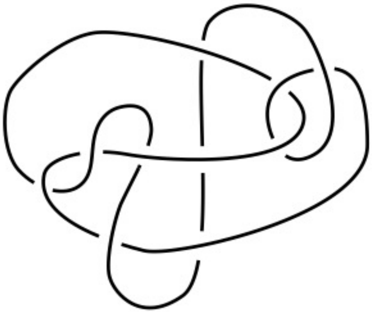
Recalcitrance of Culprit



$$C(K) = 10$$

$$Top(K) = 12$$

$$R(K) = \frac{Top(K)}{C(K)} = 1.2$$



$$C(K) = 10$$

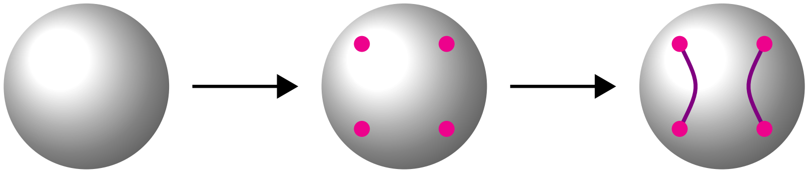
$$Top(K) = 12$$

$$R(K) = \frac{Top(K)}{C(K)} = 1.2$$

Tangles

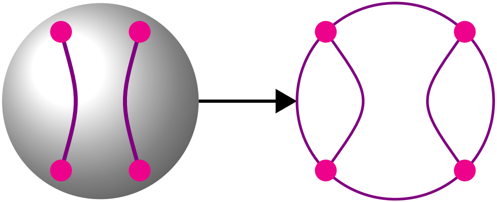
Definition 2-Tangle

A **2-tangle** is a proper embedding of two unoriented arcs and a finite number of circles in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3.

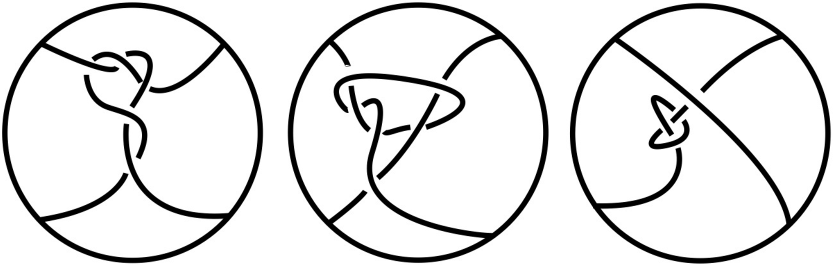


Definition Tangle Diagram

A **tangle diagram** is a regular projection of the tangle on a cross-sectional disc of B^3. By "tangle" we will mean "tangle diagram".

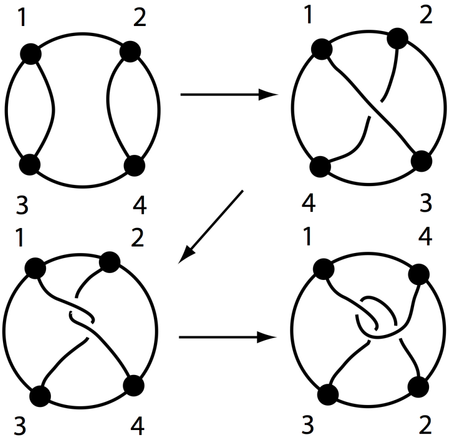


Three Types of Tangles: rational, prime, locally knotted.



Definition Rational Tangle

A **rational tangle** is a special case of a 2-tangle obtained by applying consecutive twists on neighboring endpoints of two trivial arcs.



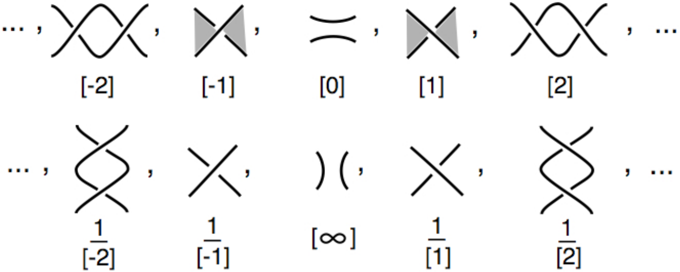


Figure 8 - The Elementary Rational Tangles and the Types of Crossings

Operations for tangles (Addition, Multiplication, and Rotation)

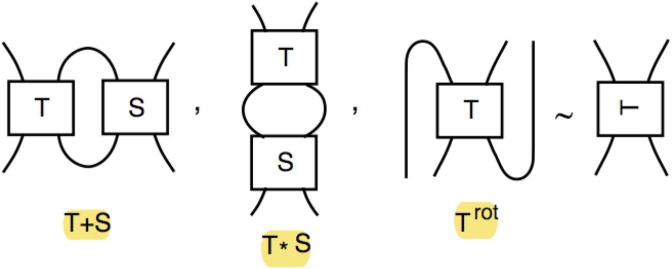


Figure 9 - Addition, Multiplication and Rotation of 2-Tangles

Note that addition and multiplication do not necessarily preserve rational tangles. Example: $\frac{1}{[3]} + \frac{1}{[3]}$. This would produce a prime tangle.

Closures of rational tangles (Numerator, Denominator)

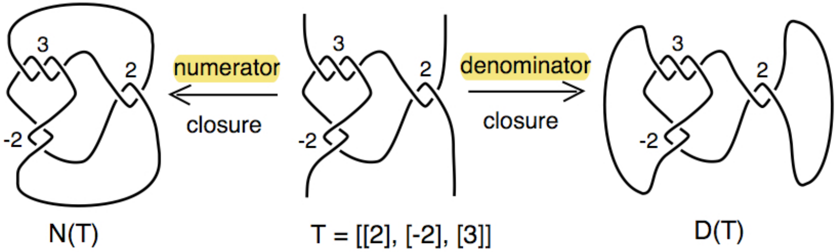


Figure 7 - A Rational Tangle and its Closures to Rational Knots

Definition Rational Knot

A **rational knot** is defined to be the numerator of a rational tangle.

Represent rational tangles in standard form.

$$T = [[a_1],[a_2],\ldots,[a_n]] = [a_1] + \cfrac{1}{[a_2] + \cfrac{1}{[a_3] + \ldots \cfrac{1}{[a_{n-1}] + \cfrac{1}{[a_n]}}}}$$

for $a_1 \in \mathbb{Z}, a_2, \ldots, a_n \in \mathbb{Z} - 0$

Fractions of Rational Tangles

$$F(T) = [a_1, a_2, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Isotopic Tangles

Definition Isotopic

Two tangles, T, S are **isotopic**, denoted by $T\ S$, if and only if they have identical configurations of their four endpoints, and they differ by a finite sequence of the Reidemeister moves which take place in the interior of the projection disc.

Theorem 1 (Conway 1975)

Theorem

Classification of Rational Tangles Two rational tangles are isotopic if and only if they have the same fraction.

Finding the Fraction of a tangle. (Take with a grain of salt.)

(1) $F([\pm 1]) = \pm 1$

(2) $F(T + S) = F(T) + F(S)$

(3) $F(T^{\text{rot}}) = -\dfrac{1}{F(T)}$

Theorem 2 (Schubert 1956)

Theorem

Suppose that rational tangles with fractions $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *are given. Here* p *and* q *are relatively prime, similarly for* p' *and* q'*. If* $K\left(\frac{p}{q}\right)$ *and* $K\left(\frac{p'}{q'}\right)$ *denote the corresponding rational knots obtained by taking numerator closures of these tangles, then* $K\left(\frac{p}{q}\right)$ *and* $K\left(\frac{p'}{q'}\right)$ *are isotopic if and only if*

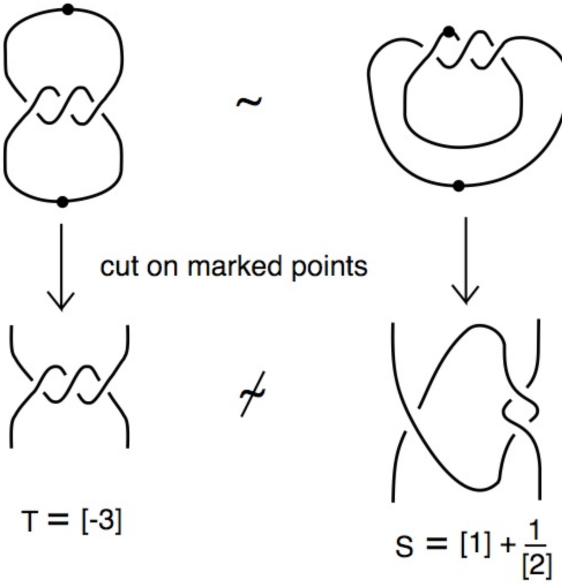
(1) $p = p'$ *and*

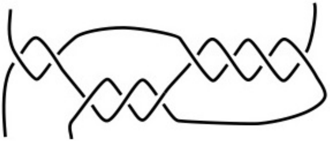
(2) *either* $q = q' \bmod p$ *or* $qq' = 1 \bmod p$*.*

Combinatorial proof of Schubert's Theorem

Given a rational knot diagram at which places may one cut so that it opens to a rational tangle?

Special Cut

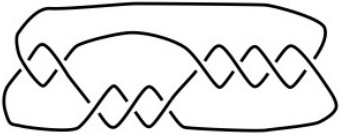




$\not\sim$

T = [2] + 1/([3] + 1/[4])

S = [4] + 1/([3] + 1/[2])



N(T) = N(S)

Theorem 3: The Palindrome Theorem

Theorem

Let $\{a_1, a_2, \ldots, a_n\}$ *be a collection of* n *integers, and let*

$$\frac{P}{Q} = [a_1, a_2, \ldots, a_n] \text{ and } \frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$$

Then $P = P'$ *and* $QQ' \equiv (-1)^{n+1} \text{mod } P$ *Moreover, for any sequence of integers* $\{a_1, a_2, \ldots, a_n\}$ *the value of the corresponding continued fraction* $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ *is given through the following matrix product*

$$M = M(a_1)M(a_2)\ldots M(a_n)$$

via the identity

$$M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$$

where this matrix also gives the evaluation of the palindrome continued fraction

Theorem 3: The Palindrome Theorem Proof Base Case

Proof

Let $\{a_1, a_2, \ldots, a_n\}$ be a collection of integers.

Base Case : Let $\frac{K}{J} = [a_1]$ and $\frac{K'}{J'} = [a_1]$.

Hence

$$\frac{K}{J} = \frac{a_1}{1} = \frac{K'}{J'}$$
$$\implies K = K'$$

and $M(a_1) = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = M^T(a_1)$

$M(a_1) = \begin{pmatrix} K & J' \\ J & H \end{pmatrix}$

where $JJ' \equiv (-1)^{n+1} \text{mod } K$ since $(1)(1) \equiv (-1)^2 \text{mod } a_1$

Proof Continued

Inductive Step : Let $\frac{R}{S} = [a_2, \ldots, a_n]$ and $\frac{R'}{S'} = [a_n, a_{n-1}, \ldots, a_2]$. By induction we can say that $N_1 = M(a_2)M(a_3)\ldots M(a_n) = \begin{pmatrix} R & S' \\ S & V \end{pmatrix}$

Note that

$$\begin{aligned} M &= M(a_1)N_1 \\ &= \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} M(a_2)M(a_3)\ldots M(a_n) \\ &= \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R & S' \\ S & V \end{pmatrix} \\ &= \begin{pmatrix} \boxed{a_1R + S} & \boxed{a_1S' + V} \\ \boxed{R} & S' \end{pmatrix} \end{aligned}$$

Proof Continued

Since

$$\frac{P}{Q} = a_1 + \frac{1}{\frac{R}{S}} = a_1 + \frac{S}{R} = \frac{a_1 R + S}{R}$$

we have that $\boxed{P = a_1 R + S}$ and $\boxed{Q = R}$

...

Now I want to show that $Q' = a_1 S' + V$.

Let $\frac{L}{W} = [a_{n-1}, \dots, a_2, a_1]$ and $\frac{L'}{W'} = [a_1, a_2, \dots, a_{n-1}]$.

$$N_2 = M(a_{n-1})M(a_{n-2})\dots M(a_1) = \begin{pmatrix} L & W' \\ W & Z \end{pmatrix}$$

$$\begin{aligned} M^T &= M(a_n)M(a_{n-1})\dots M(a_1) \\ &= M(a_n)N_2 \\ &= \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L & W' \\ W & Z \end{pmatrix} \\ &= \begin{pmatrix} a_n L + W & a_n W' + Z \\ L & W' \end{pmatrix} \end{aligned}$$

Since $\frac{P'}{Q'} = a_n + \frac{1}{\frac{L}{W}} = a_n + \frac{W}{L} = \frac{a_n L + W}{L}$

...

$$M = \begin{pmatrix} a_1R+S & \boxed{a_1S'+V} \\ R & S' \end{pmatrix}$$

$$M^T = \begin{pmatrix} a_nL+W & a_nW'+Z \\ \boxed{L} & W' \end{pmatrix}$$

Since $\dfrac{P'}{Q'} = a_n + \dfrac{1}{\frac{L}{W}} = a_n + \dfrac{W}{L} = \dfrac{a_nL+W}{L}$.

Hence $\boxed{Q' = L = a_1S'+V}$ and $M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$

Proof.

But I also want to show that $QQ' \equiv (-1)^{n+1} \text{mod } P$.

$$\begin{aligned} PU - QQ' &= \text{Det}(M) \\ &= \text{Det}(M(a_1))\text{Det}(M(a_2)) \cdots \text{Det}(M(a_n)) \\ &= \underbrace{(-1)(-1)\cdots(-1)}_{n} \\ &= (-1)^n \end{aligned}$$

□

Theorem 4

Theorem

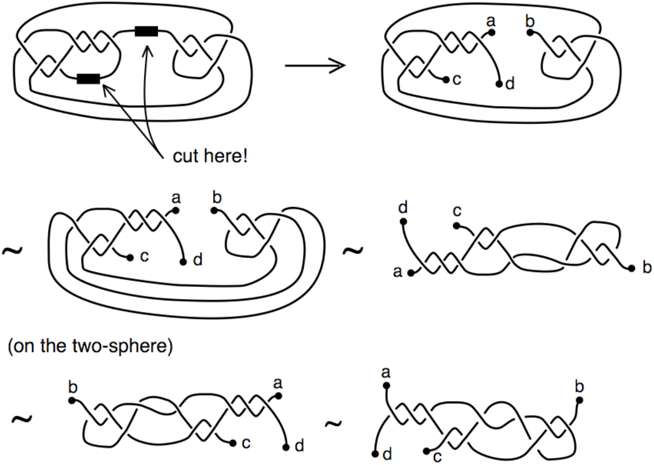
Let $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ *and* $\frac{R}{S} = [b_1, b_2, \ldots, b_m]$.

Let $A = \left[\frac{P}{Q}\right]$ *and* $B = \left[\frac{R}{S}\right]$ *be the corresponding rational tangles.*

Then the knot or link $N(A + B)$ *is rational. In fact*

$$N(A + B) = N([a_n, a_{n-1}, \ldots, a_1 + b_1, b_2, \ldots, b_m])$$

Theorem 4



N([1,2,3] + [1,1,2]) = N([3,2,1+1,1,2])

Figure 17 - The Numerator of a Sum of Rational Tangles is a Rational Link

Definition

Definition

Given continued fractions $\frac{P}{Q} = [a_1, \ldots, a_n]$ and $\frac{R}{S} = [b_1, \ldots, b_m]$, let

$$[a_1, \ldots, a_n]\#[b_1, \ldots, b_m] := [a_n, \ldots, a_2, a_1 + b_1, b_2, \ldots, b_m].$$

Theorem 5

Theorem

If $\dfrac{P}{Q}$ *has a matrix* $M = M(\vec{a}) = M(a_1)\ldots, M(a_n)$ *and* $\dfrac{R}{S}$ *has a matrix* $N = M(\vec{b}) = M(b_1), \ldots, M(b_m)$*, then* $[a_1, \ldots, a_n]\#[b_1, \ldots, b_m]$ *has matrix*

$$M\#N := M^T N^E$$

where N^E *denotes the matrix obtained by interchanging the rows of* N*. This gives an explicit formula for* $[a_1, \ldots, a_n]\#[b_1, \ldots, b_m]$*. This formula can be used to determine not only when* $N\left(\left[\dfrac{P}{Q}\right] + \left[\dfrac{R}{S}\right]\right)$ *is unknotted but also to find its knot type as a rational knot via Schubert's Theorem. In particular, we find that*

$$N\left(\left[\frac{P}{Q}\right] + \left[\frac{R}{S}\right]\right) = N\left(\left[\frac{PS+QR}{Q'S+UR}\right]\right) = N\left(\left[\frac{Num(P/Q+R/S)}{Num(Q'/U+R/S)}\right]\right)$$

where $|PU - QQ'| = 1$

WHITE BOARD

Theorem 5 Proof 1

Proof

Let $M(\vec{a}) = M(a_1) \ldots M(a_n)$ and $M(\vec{b}) = M(b_1) \ldots M(b_m)$.
By Theorem 3

$$M(\vec{a}) = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$$

$$M(\vec{b}) = \begin{pmatrix} R & S' \\ S & V \end{pmatrix}$$

Let $\dfrac{F}{G} = [a_n, a_{n-1}, \ldots, a_1 + b_1, b_2, \ldots, b_m] = \dfrac{P}{Q} \# \dfrac{R}{S}$.
Then by Theorem 4 we have

$$N\left(\left[\frac{P}{Q}\right] + \left[\frac{R}{S}\right]\right) = N\left(\left[\frac{F}{G}\right]\right)$$

and

$$M(\vec{c}) = M(a_n)M(a_{n-1}) \ldots M(a_1 + b_1)M(b_2) \ldots M(b_m)$$

Theorem 5 Proof Continued 2

Proof Continued

Note the identity

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus

$$M(\vec{c}) = M(\vec{a})^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(\vec{b}) = M(\vec{a})^T M(\vec{b})^E$$

Where M^E denotes the matrix obtained from M by interchanging its two rows. In particular, this formula implies that

$$\begin{pmatrix} F & G' \\ G & W \end{pmatrix} = \begin{pmatrix} P & Q \\ Q' & U \end{pmatrix} \begin{pmatrix} S & V \\ R & S' \end{pmatrix} = \begin{pmatrix} PS + QR & PV + QS' \\ Q'S + UR & Q'V + US' \end{pmatrix}$$

Theorem 5 Proof Continued 3

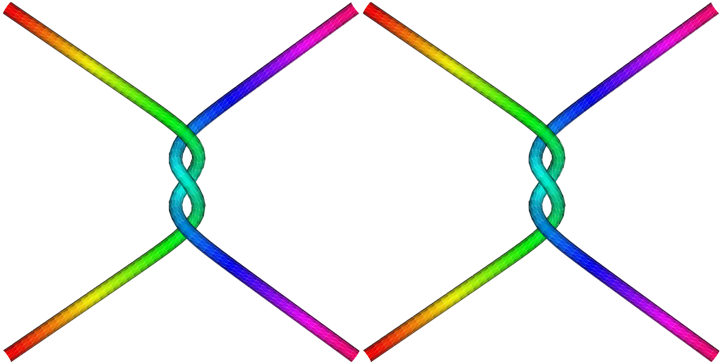
Proof.

Thus

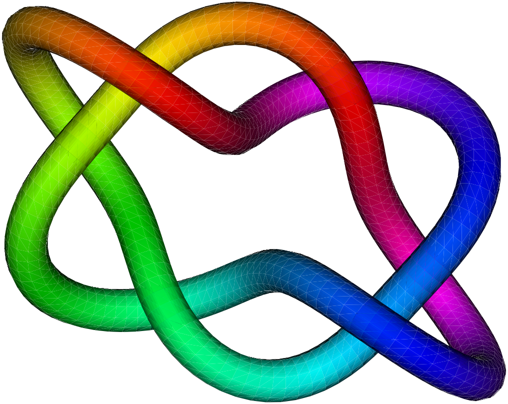
$$N\left([P/Q]+[R/S]\right)=N\left(\left[\frac{PS+QR}{Q'S+UR}\right]\right)=N\left(\left[\frac{Num(P/Q+R/S)}{Num(Q'/U+R/S)}\right]\right)$$

where $|PU-QQ'|=1$. □

Theorem 5 Example: Sum of two Rational Tangles



Theorem 5 Example: Numerator of the sum



Theorem 5 Example

Let $\vec{a} = [0, 3]$ and $\vec{b} = [0, 3]$.

$$M(\vec{a}) = M(0)M(3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$$

$$M(\vec{b}) = M(0)M(3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} R & S' \\ S & V \end{pmatrix}$$

Let $\frac{F}{G} = \frac{P}{Q} \# \frac{R}{S}$

$$N\left(\left[\frac{P}{Q}\right] + \left[\frac{R}{S}\right]\right) = N\left(\left[\frac{P}{Q} \# \frac{R}{S}\right]\right) = N\left(\left[\frac{(P)(S) + (Q)(R)}{(Q')(S) + (U)(R)}\right]\right) = N\left(\left[\frac{F}{G}\right.\right.$$

$$N\left(\frac{1}{[3]} + \frac{1}{[3]}\right) = N\left(\left[\frac{1}{3} \# \frac{1}{3}\right]\right) = N\left(\left[\frac{(1)(3) + (3)(1)}{(0)(3) + (1)(1)}\right]\right) = N([6])$$

Unknot Theorem

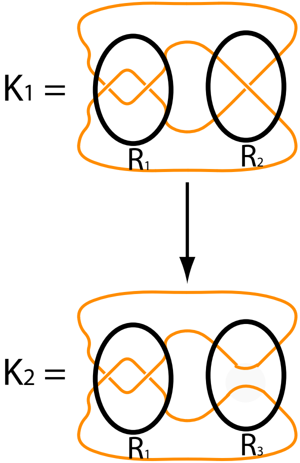
Theorem

Let $\frac{P}{Q} = [a_1, \ldots, a_n]$ *and* $\frac{R}{S} = [b_1, \ldots, b_n]$ *be as in Theorem 5. Then* $N\left(\left[\frac{P}{Q}\right] + \left[\frac{R}{S}\right]\right)$ *is unknotted if and only if* $PS + QR = \pm 1$*, that is,* PS *and* QR *are consecutive integers.*

Project Idea for Xer Recombination

$$K_1 = N(R_1 + R_2)$$
$$= N([-2] + [1])$$
$$= N\left(\left[\frac{-2}{1}\right] + \left[\frac{1}{1}\right]\right)$$
$$= N\left(\left[\frac{-2}{1} \# \frac{1}{1}\right]\right)$$
$$= N([-1])$$

unknotted



$$K_2 = N(R_1 + R_3)$$
$$= N([-2] + [0])$$
$$= N\left(\left[\frac{-2}{1}\right] + \left[\frac{0}{1}\right]\right)$$
$$= N\left(\left[\frac{-2}{1} \# \frac{0}{1}\right]\right)$$
$$= N([-2])$$

hopf link